Application of Lie algebroid structures to unification of Einstein and yang-mills field equations

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Abstract Yang-mills field equations describe new forces in the context of Lie groups and principle bundles. It is of interest to know if the new forces and gravitation can be described in the context of algebroids. This work was intended as an attempt to answer last question. The basic idea is to construct Einstein field equation in an algebroid bundle associated to space-time manifold. This equation contains Einstein and yang-mills field equations simultaneously. Also this equation yields a new equation that can have interesting experimental results.

Keywords: Lie algebra, Lie algebroid, connection, metric, curvature, gravitation, field equation, unification

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1 Introduction

Einstein field equation describes gravitational forces, and yang-mills field equations describe other forces. Principle bundles on a space-time and principle connections are main apparatus for introducing and compromising yang-mills theory and GR [1]. This method as in Kaluza-Klien theory [6],[9] make us to assume some extra dimension in space-time. Here we propose some other method that is capable of describing gravitation and new forces simultaneously and needs no extra dimension in space-time. The main idea of this method is enriching tangent bundle of the space-time by a lie algebra and make an algebroid structure. In our method, we need no extra dimension in space-time manifold, but we add some extra dimension to tangent bundle of space-time. Our approach is different from Kaluza-Klien theory,since first we have no extra dimension in space-time, second we use different mathematical structures, third our method is more general and contains results of yang-mills theory. Of course, our results is very near to that of Kaluza-Klien and yang-mills theory, but our method is different.

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In [3] we have introduced an Lie algebroid bundle that is an extension of tangent bundle of some space-time. This extension can be expressed by

$$\widehat{TM} = \sum_p T_pM \oplus I\!R = TM \oplus I\!R$$

In that structure we could unify Einstein and Maxwell equations. In that method $I\!R$ plays the role of a trivial Lie algebra. In this paper we replace $I\!R$ with an arbitrary Lie algebra $\eth$. One of the physical interpretations of this work, is replacing the unified Maxwell and Einstein field equations in [3] with the unified yang-mills and Einstein field equations.

2 Connection form and its curvature form

In this section M is a smooth manifold and $U \in \mathfrak{X}M$ and $\eth$ is a Lie algebra and $\omega \in A^1(M, \eth)$. ω has some relations to connection forms on principle bundles and may be used to define connection on some trivial vector bundles, so we call it a $\eth$-valued connection form on M(note this is just a name and it doesn't mean that ω satisfies the connection form conditions). Let W be vector space and $\rho : \eth \times W \longrightarrow W$, $\rho(h, v) = h.v$ be a Lie algebra representation of $\eth$ on W.

We can use ω to define a connection on the trivial vector bundle $M \times W$ by the following equation.

$$X \in C^\infty(M, W), \quad \nabla^\omega_U X = U(X) + \omega(U).X \tag{1}$$

By $U(X)$ we mean Lie derivation of X with respect to U. As an important example consider adjoint representation of $\eth$ on itself. If $\xi \in C^\infty(M, \eth)$, then from (1) the next equation holds.

$$\nabla^\omega_U \xi = U(\xi) + [\omega(U), \xi]$$

Proposition 2.1 *If $\xi \in C^\infty(M, \eth)$ and $X \in C^\infty(M, W)$, then we have*

$$\nabla^\omega_U(\xi.X) = (\nabla^\omega_U \xi).X + \xi.\nabla^\omega_U X \tag{2}$$

proof: Since the action of $\eth$ on W is bilinear, then

$$U(\xi.X) = U(\xi).X + \xi.U(X)$$

Now the following computations show the result.

$$\begin{aligned}
\nabla^\omega_U(\xi.X) &= U(\xi.X) + \omega(U).(\xi.X) \\
&= U(\xi).X + \xi.U(X) + \omega(U).(\xi.X) - \xi.(\omega(U).X) + \xi.(\omega(U).X) \\
&= U(\xi).X + \xi.(U(X) + \omega(U).X) + [\omega(U), \xi].X \\
&= (\nabla^\omega_U \xi).X + \xi.\nabla^\omega_U X \quad \blacksquare
\end{aligned}$$

In the case of $M \times \eth$, if $\xi, \eta \in C^\infty(M, \eth)$, then

$$\nabla^\omega_U [\xi, \eta] = [\nabla^\omega_U \xi, \eta] + [\xi, \nabla^\omega_U \eta]$$

By direct computation we find curvature tensor of ∇^ω. For $U, V \in \mathfrak{X}M$ and $X \in C^\infty(M, W)$ assuming $[U, V] = 0$, we have

$$\begin{aligned}
R^\omega(U, V)(X) &= \nabla^\omega_U \nabla^\omega_V X - \nabla^\omega_V \nabla^\omega_U X \\
&= \nabla^\omega_U (V(X) + \omega(V).X) - \nabla^\omega_V (U(X) + \omega(U).X) \\
&= UV(X) + U(\omega(V).X) + \omega(U).V(X) + \omega(U).(\omega(V).X) \\
&\quad -VU(X) - V(\omega(U).X) - \omega(V).U(X) - \omega(V).(\omega(U).X) \\
&= U(\omega(V)).X + \omega(V).U(X) + \omega(U).V(X) \\
&\quad -V(\omega(U)).X - \omega(U).V(X) - \omega(V).U(X) + [\omega(U), (\omega(V)].X \\
&= (d\omega(U, V) + [\omega(U), \omega(V)]).X
\end{aligned}$$

Define $\Omega \in A^2(M, \eth)$ as follows.

$$2\Omega(U, V) = d\omega(U, V) + [\omega(U), \omega(V)] \tag{3}$$

So,

$$R^\omega(U, V)(X) = 2\Omega(U, V).X \tag{4}$$

In the case of $M \times \eth$, if $\xi \in C^\infty(M, \eth)$, then $R^\omega(U, V)(\xi) = 2[\Omega(U, V), \xi]$. We call Ω the curvature form of ω.

Assume W has a positive definite inner product and $\eth$ acts on W anti-symmetrically i.e.

$$\forall h \in \eth,\ \forall u, v \in W, \quad < h.u, v >= - < u, h.v >$$

For example, if a Lie group acts isometrically on W, then its Lie algebra acts anti-symmetrically on W. Specially if a Lie group has a bi-invariant metric, then the adjoint representation of its Lie algebra on it self is anti-symmetric.

By this assumption, $M \times W$ is a Riemannian vector bundle and ∇^ω is a Riemannian connection i.e.

$$X, Y \in C^\infty(M, W), \quad U < X, Y >=< \nabla^\omega_U X, Y > + < X, \nabla^\omega_U Y >$$

3 Semi-Riemannian Lie algebroid $TM^\eth$

In this section M is a semi-Riemannian manifold , $U \in \mathfrak{X}M$ and $\eth$ is a Lie algebra which has an inner product and its adjoint representation action is anti-symmetric, and ω is a $\eth$- valued connection form on M.

Set $TM^\eth = \cup_{p \in M}(T_pM \oplus \eth)$. $TM^\eth$ is a vector bundle and its sections has the form $V + \xi$ in which $V \in \mathfrak{X}M$ and $\xi \in C^\infty(M, \eth)$. $TM^\eth$ has a natural Lie algebroid structure by the anchor map $\rho(V + \xi) = V$ and the following Lie bracket.

$$[U + \xi, V + \eta] = [U, V] + [\xi, \eta] + U(\eta) - V(\xi) \tag{5}$$

Straightforward computations verify that $TM^{\eth}$ is a Lie algebroid. By inner product of $\eth$ and metric of M and ω, we can define a semi-Riemannian metric on $TM^{\eth}$. As in [3] we suggest that TM is not orthogonal to $\eth$, instead some subbundle of $TM^{\eth}$ isomorphic to TM, is orthogonal to $\eth$. This subbundle is denoted by $\overline{TM}$ and defined as follows.

$$\overline{TM} = \{v + \omega(v) \mid v \in T_pM \ , \ p \in M\} \tag{6}$$

We denote $v + \omega(v)$ by $\overline{v}$.For a vector field $V \in \mathfrak{X}M$, set $\overline{V} = V + \omega(V)$. The meter of $TM^{\eth}$ is defined as follows. For $U, V \in \mathfrak{X}M$, and $\xi, \eta \in C^\infty(M, \eth)$,

$$< \overline{U} + \xi \ , \ \overline{V} + \eta >=< U, V >_M + < \xi, \eta >_{\eth} \tag{7}$$

Note that $\overline{TM}$ is not horizontal subbundle of some connections, but $\overline{TM}$ is a subbundle of $TM^{\eth}$ and is complement to trivial subbundle $M \times \eth$. In fact, by the above definition $\overline{TM}$ is orthogonal subbundle of $M \times \eth$.
Because of this definition it is better all computations be done with respect to $\overline{U}$s and ξ s. For example, brackets of these sections of $TM^{\eth}$ are computed as follows.

$$\begin{aligned} [\overline{U}, \xi] &= \nabla^\omega_U \xi \qquad (8) \\ [\overline{U}, \overline{V}] &= \overline{[U, V]} + 2\Omega(U, V) \qquad (9) \end{aligned}$$

Also, $\rho(\overline{U}) = U$. In foregoing computations we need to use some other tensors equivalent to the curvature form Ω. Ω as an operator is $\Omega : \mathfrak{X}M \times \mathfrak{X}M \longrightarrow C^\infty(M, \eth)$. we define tensor $\Omega^a : \mathfrak{X}M \times C^\infty(M, \eth) \longrightarrow \mathfrak{X}M$ as follows. For $U, V \in \mathfrak{X}M$ and $\xi \in C^\infty(M, \eth)$,

$$< \Omega^a(U, \xi), V >_M =< \Omega(U, V), \xi >_{\eth} \tag{10}$$

$\Omega^a(U, \xi)$ is anti-symmetric with respect to U.

4 Levi-civita connection and curvature of $TM^{\eth}$

In this section, M is a semi-Riemannian manifold and $U, V \in \mathfrak{X}M$, and $\eth$ is a Lie algebra which has an inner product that the action of its adjoint representation is anti-symmetric, and $\xi, \eta \in C^\infty(M, \eth)$, and ω is a $\eth$- valued connection form on M, and $TM^{\eth}$ is the semi-Riemannian Lie algebroid defined in the previous section.

Levi-civita connection of the semi-Riemannian Lie algebroid $TM^{\eth}$ is defined similar to semi-Riemannian manifolds [2]. We denote this connection by $\widehat{\nabla}$ which is defined by the following relation, if $\widehat{U}, \widehat{V}, \widehat{W}$ be arbitrary sections of $TM^{\eth}$:

$$\begin{aligned} 2 < \widehat{\nabla}_{\widehat{U}} \widehat{V}, \widehat{W} >= \ & \rho(\widehat{U}) < \widehat{V}, \widehat{W} > + \rho(\widehat{V}) < \widehat{W}, \widehat{U} > - \rho(\widehat{W}) < \widehat{U}, \widehat{V} > \\ & + < [\widehat{U}, \widehat{V}], \widehat{W} > - < [\widehat{V}, \widehat{W}], \widehat{U} > + < [\widehat{W}, \widehat{U}], \widehat{V} > \end{aligned}$$

Proposition 4.1 *Levi-civita connection of the $TM^{\eth}$ satisfies the following relations.*

$$
\begin{aligned}
\widehat{\nabla}_\xi \eta &= \frac{1}{2}[\xi,\eta] && (11)\\
\widehat{\nabla}_{\overline{U}}\xi &= -\overline{\Omega^a(U,\xi)} + \nabla^\omega_U \xi && (12)\\
\widehat{\nabla}_\xi \overline{U} &= -\overline{\Omega^a(U,\xi)} && (13)\\
\widehat{\nabla}_{\overline{U}}\overline{V} &= \overline{\nabla_U V} + \Omega(U,V) && (14)
\end{aligned}
$$

Proof: Straightforward computations show these results. For example we verify (12).

$$
\begin{aligned}
2 < \widehat{\nabla}_{\overline{U}}\xi, \eta > &= \rho(\overline{U}) < \xi,\eta > + \rho(\xi) < \eta, \overline{U} > - \rho(\eta) < \overline{U}, \xi > \\
&\quad + < [\overline{U},\xi],\eta > - < [\xi,\eta],\overline{U} > + < [\eta,\overline{U}],\xi > \\
&= U < \xi,\eta > + < \nabla^\omega_U \xi, \eta > - < \nabla^\omega_U \eta, \xi > \\
&= 2 < \nabla^\omega_U \xi, \eta >
\end{aligned}
$$

$$
\begin{aligned}
2 < \widehat{\nabla}_{\overline{U}}\xi, \overline{V} > &= \rho(\overline{U}) < \xi,\overline{V} > + \rho(\xi) < \overline{V}, \overline{U} > - \rho(\overline{V}) < \overline{U}, \xi > \\
&\quad + < [\overline{U},\xi],\overline{V} > - < [\xi,\overline{V}],\overline{U} > + < [\overline{V},\overline{U}],\xi > \\
&= < \overline{[V,U]} + 2\Omega(V,U), \xi >= 2 < \Omega^a(V,\xi), U > \\
&= -2 < \Omega^a(U,\xi), V >= -2 < \overline{\Omega^a(U,\xi)}, \overline{V} > \quad \blacksquare
\end{aligned}
$$

Proposition 4.2 *If $\theta \in C^\infty(M,\eth)$ and $W \in \mathfrak{X}M$ and $\mathbf{R}$ is the curvature tensor of M , then the curvature tensor , related to $\widehat{\nabla}$, denoted by $\widehat{\mathbf{R}}$, satisfies the following relations. The curvature tensor of $\widehat{\nabla}$, denoted by $\widehat{\mathbf{R}}$, satisfies the following relations. $\theta \in C^\infty(M,\eth)$ and $W \in \mathfrak{X}M$ and $\mathbf{R}$ is the curvature tensor of M.*

$$
\begin{aligned}
\widehat{\mathbf{R}}(\xi,\eta)(\theta) &= -\frac{1}{4}[[\xi,\eta],\theta] && (15)\\
\widehat{\mathbf{R}}(\xi,\eta)(\overline{W}) &= \overline{\Omega^a(\Omega^a(W,\eta),\xi)} - \overline{\Omega^a(\Omega^a(W,\xi),\eta)} + \overline{\Omega^a(W,[\xi,\eta])} && (16)\\
\widehat{\mathbf{R}}(\xi,\overline{V})(\theta) &= \overline{\Omega^a(\Omega^a(V,\theta),\eta)} + \frac{1}{2}\overline{\Omega^a(V,[\eta,\theta])} && (17)\\
\widehat{\mathbf{R}}(\xi,\overline{V})(\overline{W}) &= \overline{(\nabla_V\Omega^a)(W,\xi)} + \Omega(V,\Omega^a(W,\xi)) + \frac{1}{2}[\xi,\Omega(V,W)] && (18)\\
\widehat{\mathbf{R}}(\overline{U},\overline{V})(\theta) &= -\overline{(\nabla_U\Omega^a)(V,\theta)} + \overline{(\nabla_V\Omega^a)(U,\theta)} && (19)\\
&\quad -\Omega(U,\Omega^a(V,\theta)) + \Omega(V,\Omega^a(U,\theta)) + [\Omega(U,V),\theta] \\
\widehat{\mathbf{R}}(\overline{U},\overline{V})(\overline{W}) &= \overline{\mathbf{R}(U,V)(W)} + (\nabla_U\Omega)(V,W) - (\nabla_V\Omega)(U,W) && (20)\\
&\quad -\overline{\Omega^a(U,\Omega(V,W))} + \overline{\Omega^a(V,\Omega(U,W))} \\
&\quad +2\overline{\Omega^a(W,\Omega(U,V))}
\end{aligned}
$$

In the above relations, covariant derivation of Ω and Ω^a is the combination of ∇^ω and Levi-civita connection of M.

proof: Straightforward computations show these results. For example we compute (18).

$$\begin{aligned}
\widehat{\mathbf{R}}(\xi,\overline{V})(\overline{W}) &= \widehat{\nabla}_\xi\widehat{\nabla}_{\overline{V}}\overline{W} - \widehat{\nabla}_{\overline{V}}\widehat{\nabla}_\xi\overline{W} - \widehat{\nabla}_{[\xi,\overline{V}]}\overline{W}\\
&= \widehat{\nabla}_\xi(\overline{\nabla_V W} + \Omega(V,W)) - \widehat{\nabla}_{\overline{V}}(-\overline{\Omega^a(W,\xi)}) - \widehat{\nabla}_{(-\nabla^\omega_V\xi)}\overline{W}\\
&= -\overline{\Omega^a(\nabla_V W,\xi)} + \frac{1}{2}[\xi,\Omega(V,W)] + \overline{\nabla_V\Omega^a(W,\xi)} + \Omega(V,\Omega^a(W,\xi))\\
&\quad -\overline{\Omega^a(W,\nabla^\omega_V\xi)}\\
&= \overline{(\nabla_V\Omega^a)(W,\xi)} + \Omega(V,\Omega^a(W,\xi)) + \frac{1}{2}[\xi,\Omega(V,W)] \quad \blacksquare
\end{aligned}$$

To compute Ricci curvature and scalar curvature of $TM^{\eth}$, we need to consider some orthonormal basis in $\eth$ such as $\{\theta_1,\cdots,\theta_k\}$ and local orthonormal vector fields on M such as $W_1,\cdots,W_n$, in this case $\{\theta_1,\cdots,\theta_k,\overline{W_1},\cdots,\overline{W_n}\}$ is an orthonormal basis for $TM^{\eth}$. Since M is semi-Riemannian, then $< W_j, W_j >= \pm 1$ .We set $\hat{j} =< W_j, W_j >$. Also denote Killing form of $\eth$ by $\mathbf{B}$ i.e. for $h,k\in\eth$, $\mathbf{B}(h,k) = tr(\mathrm{ad}(h)\circ\mathrm{ad}(k)) = -\sum_i < [h,\theta_i],[k,\theta_i] >$. In the following we use inner products of tensors over TM and $\eth$. Note that if V and W be some inner product spaces, we can extend these inner products in the tensor spaces over V and W. For example, if $T,S : V \longrightarrow W$ be linear maps and $\{e_1\ ,\ \cdots\ ,e_n\}$ be some orthonormal base Of V, and $\hat{j} =< e_j\ ,\ e_j >$, then $< T,S >= \sum_j \hat{j} < T(e_j),S(e_j) >$. Also, If $T,S : V\times V \longrightarrow W$ be bilinear maps, then $< T,S >= \sum_{i,j}\hat{j}\hat{i} < T(e_i,e_j),S(e_i,e_j) >$. These definitions do not depend on the choice of the base.

Proposition 4.3 *The Ricci curvature tensor of $\widehat{\nabla}$, denoted by $\widehat{Ric}$, satisfies the following relations.*

$$\widehat{Ric}(\xi,\eta) = -\frac{1}{4}\mathbf{B}(\xi,\eta) + < \Omega^a(.,\xi),\Omega^a(.,\eta) > \tag{21}$$

$$\widehat{Ric}(\xi,\overline{V}) = < \mathrm{div}\Omega^a(.,\xi), V > \tag{22}$$

$$\widehat{Ric}(\overline{U},\overline{V}) = Ric(U,V) - 2 < \Omega(.,U),\Omega(.,V) > \tag{23}$$

proof: Straightforward computations show these results.

$$\begin{aligned}
\widehat{Ric}(\xi,\eta) &= \sum_i < \widehat{\mathbf{R}}(\xi,\theta_i)(\theta_i),\eta > + \sum_j \hat{j} < \widehat{\mathbf{R}}(\xi,\overline{W_j})(\overline{W_j}),\eta >\\
&= \sum_i < -\frac{1}{4}[[\xi,\theta_i],\theta_i],\eta > + \sum_j \hat{j}\Big(< \overline{(\nabla_{W_j}\Omega^a)(W_j,\xi)},\eta >\\
&\quad + < \Omega(W_j,\Omega^a(W_j,\xi)),\eta > + < \frac{1}{2}[\xi,\Omega(W_j,W_j)],\eta > \Big)\\
&= \frac{1}{4}\sum_i < [\xi,\theta_i],[\eta,\theta_i] > + \sum_j \hat{j} < \Omega^a(W_j,\xi),\Omega^a(W_j,\eta) >\\
&= -\frac{1}{4}\mathbf{B}(\xi,\eta) + < \Omega^a(.,\xi),\Omega^a(.,\eta) >
\end{aligned}$$

$$\begin{aligned}
\widehat{Ric}(\xi,\overline{V}) &= \sum_i <\widehat{\mathbf{R}}(\xi,\theta_i)(\theta_i),\overline{V}> + \sum_j \hat{j} <\widehat{\mathbf{R}}(\xi,\overline{W_j})(\overline{W_j}),\overline{V}> \\
&= \sum_i <-\frac{1}{4}[[\xi,\theta_i],\theta_i],\overline{V}> + \sum_j \hat{j}\Big(<\overline{(\nabla_{W_j}\Omega^a)(W_j,\xi)},\overline{V}> \\
&\quad + <\Omega(W_j,\Omega^a(W_j,\xi)),\overline{V}> + <\frac{1}{2}[\xi,\Omega(W_j,W_j)],\overline{V}>\Big) \\
&= \sum_j \hat{j} <(\nabla_{W_j}\Omega^a)(W_j,\xi),V> = <\mathrm{div}\Omega^a(.,\xi),V> \\
\widehat{Ric}(\overline{U},\overline{V}) &= \sum_i <\widehat{\mathbf{R}}(\overline{U},\theta_i)(\theta_i),\overline{V}> + \sum_j \hat{j} <\widehat{\mathbf{R}}(\overline{U},\overline{W_j})(\overline{W_j}),\overline{V}> \\
&= \sum_i - <\overline{\Omega^a(\Omega^a(U,\theta_i),\theta_i)} + \frac{1}{2}\overline{\Omega^a(U,[\theta_i,\theta_i])},\overline{V}> \\
&\quad \sum_j \hat{j}\Big(\overline{\mathbf{R}(U,W_j)(W_j)},\overline{V}> - <\overline{\Omega^a(U,\Omega(W_j,W_j))},\overline{V}> \\
&\quad + <\overline{\Omega^a(W_j,\Omega(U,W_j))},\overline{V}> + 2<\overline{\Omega^a(W_j,\Omega(U,W_j))},\overline{V}>\Big) \\
&= \sum_i <\Omega^a(U,\theta_i),\Omega^a(V,\theta_i)> + Ric(U,V) \\
&\quad -3\sum_j \hat{j} <\Omega(U,W_j),\Omega(V,W_j)>
\end{aligned}$$

We can prove that these two sums are equal to $<\Omega(.,U),\Omega(.,V)>$.

$$\begin{aligned}
\sum_i <\Omega^a(U,\theta_i),\Omega^a(V,\theta_i)> &= \sum_i\sum_j \hat{j} <\Omega^a(U,\theta_i),W_j><\Omega^a(V,\theta_i),W_j> \\
&= \sum_j \hat{j}\sum_i <\Omega(U,W_j),\theta_i><\Omega(V,W_j),\theta_i> \\
&= \sum_j \hat{j} <\Omega(U,W_j),\Omega(V,W_j)> = <\Omega(.,U),\Omega(.,V)>
\end{aligned}$$

So, (23) is proved.■

Proposition 4.4 *If R be the scalar curvature of M then the scalar curvature related to $\widehat{\nabla}$, denoted by $\widehat{R}$, satisfies the following relation.*

$$\widehat{R} = R - <\Omega,\Omega> - \frac{1}{4}tr(\mathbf{B}) \tag{24}$$

proof:

$$\begin{aligned}
\widehat{R} &= \sum_i \widehat{Ric}(\theta_i,\theta_i) + \sum_j \widehat{Ric}(\overline{W_j},\overline{W_j}) \\
&= \sum_i \left(-\frac{1}{4}\mathbf{B}(\theta_i,\theta_i) + < \Omega^a(.,\theta_i), \Omega^a(.,\theta_i) > \right) \\
&\quad + \sum_j (Ric(W_j,W_j) - 2 < \Omega(.,W_j), \Omega(.,W_j) >) \\
&= -\frac{1}{4}tr(\mathbf{B}) + R + < \Omega^a, \Omega^a > -2 < \Omega, \Omega >
\end{aligned}$$

Since $< \Omega^a, \Omega^a > = < \Omega, \Omega >$, the proposition is proved. ■

5 Application to general relativity and yang-mills theory

In this section M is a space-time, and g and ω are same as previous section. We can interpret ω as potential for new relativistic forces in a similar way that yang-mills theory describes. In this interpretation, geodesics of $TM^{\eth}$ must show path of point particles influenced by gravitation and new forces.

Definition 5.1 *A smooth curve $\widehat{\alpha} : I \longrightarrow TM^{\eth}$ is called a velocity-curve whenever there exist a curve $\alpha : I \longrightarrow M$ such that $\rho(\widehat{\alpha}) = \alpha'$.*

In this case, there exists a curve $\xi : I \longrightarrow \eth$ such that $\widehat{\alpha}(t) = \overline{\alpha'(t)} + \xi(t)$. A smooth map $\widehat{\beta} : I \longrightarrow TM^{\eth}$ is called along the velocity-curve $\widehat{\alpha}$, whenever $\rho(\widehat{\beta}(t)) \in T_{\alpha(t)}M$ i.e. $\pi \circ \rho \circ \widehat{\alpha} = \pi \circ \rho \circ \widehat{\beta}$. The covariant derivation of a map along a velocity-curve $\widehat{\alpha}(t))$ is definable. A velocity-curve $\widehat{\alpha}$ is called geodesic whenever $\widehat{\nabla}_{\widehat{\alpha}}\widehat{\alpha} = 0$.

Proposition 5.2 *A velocity-curve $\widehat{\alpha}(t) = \overline{\alpha'(t)} + \xi(t)$ is geodesic iff $\nabla_{\alpha'(t)}\alpha'(t) = 2\Omega^a(\alpha'(t), \xi(t))$ and $\xi(t)$ is parallel along $\alpha(t)$ with respect to ∇^{ω}.*

Proof:

$$\begin{aligned}
\widehat{\nabla}_{\widehat{\alpha}}\widehat{\alpha} &= \widehat{\nabla}_{\overline{\alpha'(t)}+\xi(t)}\overline{\alpha'(t)} + \xi(t) \\
&= \widehat{\nabla}_{\overline{\alpha'(t)}}\overline{\alpha'(t)} + \widehat{\nabla}_{\overline{\alpha'(t)}}\xi(t) + \widehat{\nabla}_{\xi(t)}\overline{\alpha'(t)} + \widehat{\nabla}_{\xi(t)}\xi(t) \\
&= \overline{\nabla_{\alpha'(t)}\alpha'(t)} - \overline{\Omega^a(\alpha'(t),\xi(t))} + \nabla^{\omega}_{\alpha'(t)}\xi(t) - \overline{\Omega^a(\alpha'(t),\xi(t))}
\end{aligned}$$

Therefore, $\widehat{\nabla}_{\widehat{\alpha}}\widehat{\alpha} = 0$ iff $\nabla_{\alpha'(t)}\alpha'(t) = 2\Omega^a(\alpha'(t), \xi(t))$ and $\nabla^{\omega}_{\alpha'(t)}\xi(t) = 0$. ■

In the case $\eth = I\!R$, ξ represents ratio of charge to mass and $2\Omega^a(\alpha'(t), \xi(t))$ represents electromagnetic force exerted to the particle [3]. So we can consider elements of $\eth$ as vector charges (divided by mass) and $2\Omega^a(\alpha'(t), \xi(t))$ can be considered as the force produced by these vector charges.

To find more detailed and explicit information of new forces and their analogy to electromagnetism, it is better to write connection form ω and its curvature Ω with respect to the base $\{\theta_1, \cdots, \theta_k\}$.

$$\begin{array}{rcll} \omega & = & \omega^i \theta_i & \omega^i \in A^1(M) \quad i = 1, \cdots, k \\ \Omega & = & \Omega^i \theta_i & \Omega^i \in A^2(M) \quad i = 1, \cdots, k \end{array}$$

Suppose $[\theta_i, \theta_j] = C_{ij}^l \theta_l$, since $2\Omega(U, V) = d\omega(U, V) + [\omega(U), \omega(V)]$, by computation we find:

$$2\Omega^l = d\omega^l + C_{ij}^l \omega^i \wedge \omega^j \tag{25}$$

Set $\Omega_i^a(U) = \Omega^a(U, \theta_i)$. Ω_i^a is the 1-1-form equivalent to 2-form Ω^i.

Corollary 5.3 *Let $\widehat{\alpha}(t) = \overline{\alpha'(t)} + \xi(t)$ be a geodesic and $\xi(t) = \xi^i(t)\theta_i$, then*

$$\nabla_{\alpha'(t)} \alpha'(t) = 2\xi^l(t) \Omega_l^a(\alpha'(t)) \tag{26}$$

$$\xi^{l\prime}(t) = C_{ij}^l \xi^i(t) \omega^j(\alpha'(t)) \tag{27}$$

We can interpret ξ^l as $l-$th charge and Ω_l^a as $l-$th electromagnetism field and $2\xi^l(t)\Omega_l^a(\alpha'(t))$ as sum of the the forces exerted by these fields. Of course these forces are not independent and (25) shows these forces are dependent to each other and are components of a more general force.

Set $\mathbf{B}_{ij} = \mathbf{B}(\theta_i, \theta_j)$. $\widehat{Ric}$ and $\widehat{R}$ with respect to the base $\{\theta_1, \cdots, \theta_k\}$, can be written as follows:

$$\widehat{Ric}(\theta_i, \theta_j) = -\frac{1}{4}\mathbf{B}_{ij} + < \Omega_i^a, \Omega_j^a > \tag{28}$$

$$\widehat{Ric}(\theta_i, \overline{V}) = < \mathrm{div}\Omega_i^a, V > \tag{29}$$

$$\widehat{Ric}(\overline{U}, \overline{V}) = Ric(U, V) - 2\sum_i < \Omega_i^a(U), \Omega_i^a(V) > \tag{30}$$

$$\widehat{R} = R - \sum_i < \Omega_i^a, \Omega_i^a > -\frac{1}{4}\sum_i \mathbf{B}_{ii} \tag{31}$$

In the case of $\eth = I\!R$, energy-momentum tensor of electromagnetism forces in a suitable system of measurement,$(c = 1\ ,\ G = 1\ ,\ \epsilon_0 = \frac{1}{16\pi})$[3], is defined as follows.

$$\mathbf{T}^{elec} = \frac{1}{4\pi}(< \Omega^a(.), \Omega^a(.) > -\frac{1}{4} < \Omega^a, \Omega^a > \mathbf{g}) \tag{32}$$

$\mathbf{g}$ is the tensor metric of M. We can extend this definition and define $i-$th energy-momentum tensor of ω by

$$\mathbf{T}_i^\omega = \frac{1}{4\pi}(< \Omega_i^a(.), \Omega_i^a(.) > -\frac{1}{4} < \Omega_i^a, \Omega_i^a > \mathbf{g}) \tag{33}$$

Define the whole energy-momentum tensor of ω by $T^\omega = \sum_i T_i^\omega$. This definition dose not depend on the choice of θ_i, in fact:

$$\mathbf{T}^\omega(U, V) = \frac{1}{4\pi}(< \Omega^a(U, .), \Omega^a(V, .) > -\frac{1}{4} < \Omega^a, \Omega^a >< U, V >) \tag{34}$$

Denote the meter of $TM^{\eth}$ by $\widehat{\mathbf{g}}$. To construct a suitable field equation that produces Einstein field equation, we should imitate Einstein field equation in the context of this algebroid bundle. So by analogy, we can consider $\widehat{G} = \widehat{Ric} - \frac{1}{2}\widehat{R}\widehat{\mathbf{g}}$ as the extended Einstein tensor. This tensor in block form looks like the following matrix.

$$\left(\begin{array}{c|c} Ric - \frac{1}{2}R\mathbf{g} + \frac{1}{8}\mathrm{tr}(\mathbf{B})\mathbf{g} - 8\pi T^{\omega} & \mathrm{div}\Omega^a \\ \hline {}^t\mathrm{div}\Omega^a & \lambda_{ij} \end{array}\right)$$

$$\lambda_{ij} = -\frac{1}{4}\mathbf{B}_{ij} + < \Omega_i^a, \Omega_j^a > -\frac{1}{2}\widehat{R}\delta_{ij}$$

Note that $\widehat{R} = R - < \Omega, \Omega > -\frac{1}{4}\mathrm{tr}(\mathbf{B})$. Now, we can construct vacuum field equation as follows.

$$\widehat{Ric} - \frac{1}{2}\widehat{R}\widehat{\mathbf{g}} = 0 \tag{35}$$

This equation yields $\widehat{R} = 0$ and is equivalent to $\widehat{Ric} = 0$ and has the following consequences.

$$Ric - \frac{1}{2}R\mathbf{g} + \frac{1}{8}\mathrm{tr}(\mathbf{B})\mathbf{g} = 8\pi T^{\omega} \tag{36}$$

$$\mathrm{div}\Omega_i^a = 0 \tag{37}$$

$$< \Omega_i^a, \Omega_j^a > = \frac{1}{4}\mathbf{B}_{ij} \tag{38}$$

Note that in λ_{ij}, we have $\widehat{R} = 0$, so $\lambda_{ij} = 0$ yields (38). Two of these equations are Einstein and yang-mills field equations in vacuum and we find a third new equation, (38), that may have new results. Moreover, Einstein field equation naturally yields a cosmological constant that depends on inner product of $\eth$ and by re-scaling can be adapted its value to experimental data.

Particles are modeled by representations of the Lie algebra $\eth$. Suppose a representation of $\eth$ on some inner product vector spaces W that $\eth$ acts on W anti-symmetrically. Any $X \in C^{\infty}(M, W)$ is called a particle field. We can consider $\rho =< X, X >$ as density of this particle field. Charge density of a particle field can be considered as a smooth function $\eta : M \longrightarrow \eth$.

In order to construct the field equation including matter, we should extend the concept of energy-momentum tensor of matter, and we do this the same as [3]. Let T^{mass} be the ordinary energy-momentum tensor of the particle field. For every observer Z, $T^{\mathrm{mass}}(Z, Z)$ is the energy of the particle measured by Z[8]. We can define current of this particle field to be $J \in A^1(M, \eth)$ such that for every observer Z, $J(Z)$ is the vector charge of the particle field measured by Z.

T^{mass} and J are parts of the extended energy-momentum tensor which is denoted by $\widehat{T}$. In fact for $U, V \in \mathfrak{X}M$ and $\xi \in C^{\infty}(M, \eth)$, we define:

$$\widehat{T}(\bar{U}, \bar{V}) = T^{\mathrm{mass}}(U, V) \quad , \quad \widehat{T}(\bar{U}, \xi) =< J(U), \xi >$$

To complete the construction of $\widehat{T}$, similar to [3], we need a symmetric 2-tensor on $\eth$. It seems we should consider $\frac{1}{\rho}\eta \otimes \eta$ for this tensor. Because, in the case

$\eth = I\!R$ in [3] we have good reason for it and it is very natural. Of course, only results of this choice and experience can show that this choice is true or not. So we propose, $\widehat{T}(\xi_1, \xi_2) = \frac{1}{\rho} < \xi_1, \eta >< \xi_2, \eta >$, and $\widehat{T}$ be defined as follows:

$$\widehat{T} = \left(\begin{array}{c|c} T^{\text{mass}} & {}^tJ \\ \hline J & \frac{1}{\rho}\eta \otimes \eta \end{array} \right) \tag{39}$$

Now, we can write the Einstein field equation in this structure as following.

$$\widehat{Ric} - \frac{1}{2}\widehat{R}\widehat{\mathbf{g}} = 8\pi\widehat{T} \tag{40}$$

This equation contains three following equations.

$$Ric - \frac{1}{2}R\mathbf{g} + \frac{1}{8}\text{tr}(\mathbf{B})\mathbf{g} = 8\pi(T^\omega + T^{\text{mass}}) \tag{41}$$

$${}^t\text{div}\Omega = 8\pi J \tag{42}$$

$$\lambda_{ij} = 8\pi\frac{\eta_i\eta_j}{\rho} \tag{43}$$

The first two equations are Einstein and yang-mills field equations and third equation is new and may have new results. Equation (40) makes Einstein and yang-mills theory into a unified theory in the context of lie algebroid structures.

6 conclusion

These constructions retrieve yang-mills theory in the context of lie algebroid structures and they need no principle bundle and principle connection. This theory is more simple and does not make any extra dimension in space-time, instead it enriches tangent bundle by a lie algebra and makes a lie algebroid and replaces tangent bundle by this lie algebroid.

Of course, this theory does not contain quantum effects and internal structures of particles. This theory must be improved such that internal structures of particles determine density and vector charge density naturally.

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